10.6. CROUT'S METHOD

We shall explains this method by considering three equations. Let us consider the equation as follows :

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} . \qquad ...(1)$$

The argumented matrix of (1) is

$$(A \mid B) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{vmatrix} . \qquad ...(2)$$

Now, we consider derived matrix as follows :

$$(A' \mid B') = \begin{vmatrix} a'_{11} & a'_{12} & a'_{13} & b'_1 \\ a'_{21} & a'_{22} & a'_{23} & b'_2 \\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{vmatrix} . \qquad ...(3)$$

Step I. First column of (3) is same as first column of (2)

i.e.,
$$a'_{11} = a_{11}, a'_{21} = a_{21}, a'_{31} = a_{31}$$

or
$$a'_{i1} = a_{i1} \text{ for all } i = 1, 2, 3.$$

Step II. Elements of first row to the right of first column in (3) are given by

$$a'_{12} = \frac{a_{12}}{a_{11}}, a'_{13} = \frac{a_{13}}{a_{11}}, b'_1 = \frac{b_1}{a_{11}}$$

i.e.,
$$a'_{1j} = \frac{a_{1j}}{a_{11}}, j = 2, 3.$$

Step III. Elements of second column except a'_{12} are given by

$$a'_{22} = a_{22} - a'_{12}a'_{21}$$

$$a'_{32} = a_{32} - a'_{12}a'_{31}$$

i.e.,
$$a'_{j2} = a_{j2} - a'_{12}a'_{j1}, j = 2, 3.$$

Step IV. Elements of second row except a'_{21}, a'_{22} are given by

$$a'_{23} = \frac{a_{23} - a'_{13}a'_{21}}{a'_{22}}, \quad b_2' = \frac{b_2 - b_1'a'_{21}}{a'_{22}}$$

i.e.,
$$a'_{2j} = \frac{a_{2j} - a'_{1j}a'_{21}}{a'_{22}}, j = 3.$$

Step V. Elements of third column except a'_{13}, a'_{23} is given by

$$a'_{33} = a_{33} - a'_{31}a'_{13} - a'_{32}a'_{23}.$$

Step VI. Element of third row except $a'_{31}, a'_{32}, a'_{33}$ is

$$b'_3 = \frac{b_3 - b'_2 a'_{32} - b'_1 a'_{31}}{a'_{33}}.$$

Thus, the solution of the given equations is given by

$$x_3 = b'_3,\ x_2 = b'_2 - a'_{23}x_3,\ x_1 = b'_1 - a'_{13}x_3 - a'_{13}x_2.$$

This Crout's method is explained properly below in which the coefficient a' and constants b' are discussed how they are obtained.

Crout established a method in which Gauss elimination is often performed. Let us consider a system of three equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array}\right\}. \qquad ...(1)$$

Above equations can be written as

$$AX = B$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad ...(2)$$

$\therefore$ Augmented matrix

$$(A \mid B) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{vmatrix}. \qquad ...(3)$$

Equation (1) becomes after Gauss elimination process as follows :

$$\left.\begin{array}{r} x_1 + a'_{12}x_2 + a'_{13}x_3 = b_1' \\ x_2 + a'_{23}x_3 = b_2' \\ x_3 = b_3' \end{array}\right\}. \qquad ...(4)$$

Now the first equation of (1) is obtained by multiplication of first equation in (4) by a constant a'_{11}, and second equation of (1) is obtained through multiplication of first and second equation in (4) by a'_{21} and a'_{22} respectively, and adding. Similarly, the third equation of (1) is obtained through multiplication of first, second and third equations in (4) by a'_{31}, a'_{32} and a'_{33} and then adding. Thus, we get the following equations

$$\left.\begin{array}{r} a'_{11}b_1' = b_1 \\ a'_{21}b_1' + a'_{22}b_2' = b_2 \\ a'_{31}b_1' + a'_{32}b_2' + a'_{33}b_3' = b_3 \end{array}\right\}. \qquad ...(5)$$

Let us introduce the matrices P and Q as follows :

$$P = \begin{bmatrix} a'_{11} & 0 & 0 \\ a'_{21} & a'_{22} & 0 \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix},\ Q = \begin{bmatrix} 0 & a'_{12} & a'_{13} \\ 0 & 0 & a'_{33} \\ 0 & 0 & 0 \end{bmatrix}$$

or $$P + Q = A' \text{ (say)}.$$

Now equation (1), (4) and (5) take the form

$$\left.\begin{array}{r} AX = B \\ (Q + I)X = B' \\ PB' = B \end{array}\right\}. \qquad ...(6)$$

From (6), we obtain $P(Q+I)X = AX$

and hence $P(Q+I) = A.$

Augmenting the matrix $(Q+I)$ with new column B' and augmenting the matrix A by column B. Thus, we get

$$\begin{bmatrix} a'_{11} & 0 & 0 \\ a'_{21} & a'_{22} & 0 \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix} \begin{bmatrix} 1 & a'_{12} & a'_{13} & b'_1 \\ 0 & 1 & a'_{23} & b'_2 \\ 0 & 0 & 1 & b'_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}.$$

From above equations, we get

$$a'_{11} = a_{11},\ a'_{21} = a_{21},\ a'_{31} = a_{31}$$

$$a'_{12} = \frac{a_{12}}{a'_{11}} = \frac{a_{12}}{a_{11}},\ a'_{13} = \frac{a_{13}}{a'_{11}} = \frac{a_{13}}{a_{11}},\ b'_1 = \frac{b_1}{a_{11}}$$

$$a'_{22} = a_{22} - a'_{21} \cdot a'_{12},\ a'_{23} = \frac{1}{a'_{22}}(a_{23} - a'_{21}a'_{13})$$

$$a'_{21}b'_1 + a'_{22}b'_2 = b_2$$

or $$b'_2 = \frac{b_2 - a'_{21}b'_1}{a'_{22}}$$

$$a'_{31}a'_{12} + a'_{32} = a_{32}$$

or $$a'_{32} = a_{32} - a'_{31}a'_{12}$$

$$a'_{31}a'_{13} + a'_{32}a'_{23} + a'_{33} = a_{33}$$

or $$a'_{33} = a_{33} - a'_{31}a'_{13} - a'_{32}a'_{23}$$

$$a'_{31}b'_1 + a'_{32}b'_2 + a'_{33}b'_3 = b_3$$

or $$b'_3 = \frac{b_3 - a'_{31}b'_1 - a'_{32}b'_2}{a'_{33}}.$$

Thus, after getting all a' and b', with the help of (4), we get the solution, given by

$$x_3 = b'_3,\ x_2 = b'_2 - a'_{23}x_3,\ x_1 = b'_1 - a'_{12}x_2 - a'_{13}x_3.$$

SOLVED EXAMPLES

Example 1. *Solve the following equations by Crout's method :*

$$x_1 + x_2 + x_3 = 1$$
$$3x_1 + x_2 - 3x_3 = 5$$
$$x_1 - 2x_2 - 5x_3 = 10.$$

Solution. Above equation can be written as

$$AX = B \qquad ...(1)$$

where $$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}.$$

$$\therefore \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & -3 \\ 1 & -2 & -5 \end{bmatrix}.$$

Now derived matrix is given by

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & b'_1 \\ a'_{21} & a'_{22} & a'_{23} & b'_2 \\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{bmatrix}$$

where $a'_{11} = a_{11} = 1, a'_{21} = a_{21} = 3, a'_{31} = a_{31} = 1$

and $a'_{12} = \frac{a_{12}}{a_{11}} = \frac{1}{1} = 1, a'_{13} = \frac{a_{13}}{a_{11}} = \frac{1}{1} = 1, b'_1 = \frac{b_1}{b_{11}} = \frac{1}{1} = 1$

and $a'_{22} = a_{22} - a'_{12} \cdot a'_{21} = 1 - 1 \,.\, 3 = 1 - 3 = -2$

$$a'_{32} = a_{32} - a'_{12} \cdot a'_{31} = -2 - 1\,(1) = -3$$

$$a'_{23} = \frac{a_{23} - a'_{13}a'_{21}}{a'_{22}} = \frac{-3-3}{-2} = 3$$

$$b'_2 = \frac{b_2 - b'_{21}b'_1}{a'_{22}} = \frac{5-3}{-2} = -1$$

$$a'_{33} = a_{33} - a'_{31}a'_{13} - a'_{32}a'_{23} = -5 - 1 + 9 = 3$$

and $b'_3 = \frac{b_3 - a'_{31}b'_1 - a'_{32}b'_2}{a'_{33}} = \frac{10-1-3}{3} = \frac{6}{3} = 2.$

Thus, the solution is

$$x_3 = b'_3 = 2$$

and $x_2 = b'_2 - a'_{23}x_3 = -1 - 3\,(2) = -7$

and $x_1 = b'_1 - a'_{12}x_2 - a'_{13}x_3$

$$= 1 - 1\,(-7) - 1\,(2) = 1 + 7 - 2 = 6.$$

Hence $x_1 = 6, x_2 = -7, x_3 = 2.$

Example 2. *Solve the following equations by Crout's method :*

$$2x_1 + 3x_2 + x_3 = -1$$
$$5x_1 + x_2 + x_3 = 9$$
$$3x_1 + 2x_2 + 4x_3 = 11.$$

Solution. Above equations can be written as

$$AX = B$$

where $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}.$

Now derived matrix is given by

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & b'_1 \\ a'_{21} & a'_{22} & a'_{23} & b'_2 \\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{bmatrix}$$

and

$$(A \mid B) = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 5 & 1 & 1 & 9 \\ 3 & 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}.$$

Now find a'_{ij} as follows :

$$a'_{11} = a_{11} = 2, \; a'_{21} = a_{21} = 5, \; a'_{31} = a_{31} = 3$$

$$a'_{12} = \frac{a_{12}}{a_{11}} = \frac{3}{2}, \quad a'_{13} = \frac{a_{13}}{a_{11}} = \frac{1}{2}, \quad b'_1 = \frac{b_1}{a_{11}} = \frac{-1}{2}$$

$$a'_{22} = a_{22} - a'_{12}a'_{21}$$

$$= 1 - \frac{3}{2}(5) = 1 - \frac{15}{2} = -\frac{13}{2}$$

$$a'_{32} = a_{32} - a'_{12}a'_{31}$$

$$= 2 - \frac{3}{2}(3) = 2 - \frac{9}{2} = -\frac{5}{2}$$

$$a'_{23} = \frac{a_{23} - a'_{13}a'_{21}}{a'_{22}} = \frac{1 - \frac{1}{2}(5)}{-\frac{13}{2}} = \frac{3}{13}$$

$$b'_2 = \frac{b_2 - b'_{21}b'_1}{a'_{22}} = \frac{9 - 5\left(-\frac{1}{2}\right)}{-\frac{13}{2}} = \frac{\frac{23}{2}}{-\frac{13}{2}} = -\frac{23}{13}$$

$$a'_{33} = a_{33} - a'_{31}a'_{13} - a'_{32}a'_{23}$$

$$= 4 - 3\left(\frac{1}{2}\right) - \left(-\frac{5}{2}\right)\left(\frac{3}{13}\right) = 4 - \frac{3}{2} + \frac{15}{26}$$

$$= \frac{104 - 39 + 15}{26} = \frac{80}{26} = \frac{40}{13}$$

$$b'_3 = \frac{b_3 - a'_{31}b'_1 - a'_{32}b'_2}{a'_{33}}$$

$$= \frac{11 - 3\left(-\frac{1}{2}\right) - \left(-\frac{5}{2}\right)\left(-\frac{23}{13}\right)}{\frac{43}{13}}$$

$$= \frac{11 + \frac{3}{2} - \frac{115}{26}}{\frac{40}{13}} = \frac{\frac{(286 + 39 - 115)}{26}}{\frac{40}{13}} = \frac{\frac{210}{26}}{\frac{40}{13}} = \frac{210}{80}$$

$$b'_3 = \frac{21}{8}$$

Thus solution is

$$x_3 = b'_3 = \frac{21}{8}$$

and $$x_2 = b'_2 - a'_{23}x_3$$

$$= -\frac{23}{13} - \left(\frac{3}{13}\right)\frac{21}{8} = -\frac{19}{8}$$

and $$x_1 = b'_1 - a'_{12}x_2 - a'_{13}x_3$$

$$= -\frac{1}{2} - \frac{3}{2}\left(-\frac{19}{8}\right) - \frac{1}{2}\left(\frac{21}{8}\right)$$

$$= -\frac{1}{2} + \frac{57}{16} - \frac{21}{16} = \frac{-8+57-221}{16}$$

$$= \frac{14}{8} = \frac{7}{4}.$$

Hence, the solution is

$$x_1 = \frac{7}{4},\ x_2 = -\frac{19}{8},\ x_3 = \frac{21}{8}.$$